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'True' Self-Avoiding Walks with Generalized Bond Repulsion on $\mathbb{Z}$

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We consider a nearest-neighbor random walk on $\mathbb{Z}$, for which the probability of jumping along a bond of the lattice is proportional to $\exp[-g \cdot (\text{number of previous jumps along that bond})^\kappa]$, with $g > 0$, $\kappa \in (0, 1]$. After a review of earlier results obtained for the case $\kappa = 1$ we outline the generalizations for $\kappa \in (0, 1)$, obtaining a whole range of anomalous diffusion limits.

KEY WORDS: Limit theorems; anomalous diffusion; self-repulsive random walk.

1. INTRODUCTION

We consider 'true' self-avoiding random walks (TSAW) on $\mathbb{Z}$, with generalized bond repulsion. X_i is a random walk on the one-dimensional integer lattice starting from the origin and at time $i+1$ it jumps to one of the two neighboring sites of X_i, so that the probability of jumping along a bond of the lattice is proportional to

$$\exp[-g \cdot (\text{number of previous jumps along that bond})^\kappa]$$

where g is a positive coupling constant and $\kappa \in (0, 1]$. More formally: for a nearest neighbor walk $\underline{x}_0^i = (x_0, x_1, ..., x_i)$ and a lattice site $y \in \mathbb{Z}$ denote by $v(y \mid \underline{x}_0^i)$ the number of jumps along the bond $(y-1, y)$ performed by the walk $\underline{x}_0^i$:

$$v(y \mid \underline{x}_0^i) = \#\{0 \leqslant j < i: \{x_j, x_{j+1}\} = \{y-1, y\}\} \tag{1.1}$$

Dedicated to Oliver Penrose on the occasion of his 65th birthday.

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(Note that both left and right jumps are counted.) The walk is governed by the law

$$\mathbf{P}(X_{i+1} = x_i \pm 1 \mid \underline{X}_0^i = \underline{x}_0^i) = \frac{\exp\{-g v^\kappa(x_i + (1 \pm 1)/2 \mid \underline{x}_0^i)\}}{\exp\{-g v^\kappa(x_i + 1 \mid \underline{x}_0^i)\} + \exp\{-g v^\kappa(x_i \mid \underline{x}_0^i)\}} \tag{1.2}$$

The case $\kappa = 1$ is usually referred to as 'true' self-avoiding walk with bond repulsion. The difference from the 'orthodox' true self-avoiding walk with site repulsion studied by Amit *et al.*[1] is that here we count the local time spent on edges, while in case of site repulsion the jump probabilities are determined by the local time spent on sites. We expect that the physical phenomena, such as recurrence or (anomalous) diffusion rate, should be similar in the two cases. Based on a renormalization group analysis Amit *et al.* argued that the upper critical dimension of TSAW is $d_c = 2$. That is: in more than two dimensions the TSAW behaves diffusively, like an ordinary random walk, with logarithmic corrections in two dimensions. Computer simulations of the same authors seem to agree with this conjecture. It is natural to expect superdiffusive behaviour below the critical dimension, i.e., in $d = 1$. Some years later, Peliti and Pietronero[10] considered the one-dimensional problem, too, and based on nonrigorous scaling arguments they concluded that the variance of the TSAW should be $\sim t^{4/3}$ for long times. For a review of the problem see also refs. 7 and 9. In ref. 12 we considered the $\kappa = 1$ case and gave a rigorous version of this last assertion, proving limit theorems for the distributions of various functionals of the TSAW. Our main motivation for considering the generalization to $\kappa \in (0, 1)$ is to 'interpolate' between the $t^{1/2}$ scaling of the ordinary random walk and the anomalous $t^{2/3}$ scaling of the TSAW found in ref. 12.

The outline of the paper is the following: in Section 2 we review, without proofs, our earlier results regarding the $\kappa = 1$ case. This section contains also the definitions and notations used in the sequel. In Section 3 we formulate results and conjectures regarding the cases $\kappa \in (0, 1)$, which parallel those presented in Section 2. (As a consequence Sections 2 and 3 have a similar structure.) Finally, in Section 4 we sketch the main ideas of the proofs of the results formulated in Section 3, with emphasis on those points which differ from the proofs in refs. 12. We do not repeat the technical parts from the proofs in those papers.

2. $\kappa = 1$: REVIEW OF EARLIER RESULTS

In the present section we first review the results proved in ref. 12 concerning the local time process, hitting times, and the scaling limit of the TSAW with bond repulsion on $\mathbb{Z}$.

Let $|W_y|$, $y \in (-\infty, \infty)$, be a (two-sided) reflected Brownian motion with an arbitrary starting point $|W_0| = h \in [0, \infty)$. For $x \in [0, \infty)$ define

$$\omega_x^- = \sup\{y < x : |W_y| = 0\}, \qquad \omega_x^+ = \inf\{y > x : |W_y| = 0\} \tag{2.1}$$

and

$$T_x = \int_{\omega_0^-}^{\omega_x^+} |W_y| \, dy \tag{2.2}$$

That is, ω_x^-, respectively ω_x^+, is the starting, respectively ending, point of the excursion stradling $x \in [0, \infty)$, and T_x is the area under $|W_y|$ between ω_0^- and ω_x^+.

2.1. The Local Time Process and Hitting Times

Our first result was a limit theorem for the local time process of the TSAW X_i, stopped at appropriately defined stopping times. Let $k \in \mathbb{Z} \cap (0, \infty)$ and $m \in \mathbb{N}$. Denote by $T_{k,m}^{(L)}$ and $T_{k,m}^{(R)}$ the times of the $(m+1)$th arrival at the lattice site k from $k-1$, respectively $k+1$:

$$T_{k,-1}^{(L,R)} = 0 \tag{2.3}$$

$$T_{k,m+1}^{(L,R)} = \min\{i > T_{k,m}^{(L,R)} : X_{i-1} = k \mp 1,\ X_i = k\} \tag{2.4}$$

The bond local time process of the TSAW stopped at $T_{k,m}^{(*)}$, $* = L$ or R, is

$$S_{k,m}^{(*)}(l) = v(k - l \,|\, \underline{X}_0^{T_{k,m}^{(*)}}), \qquad l \in \mathbb{Z} \tag{2.5}$$

Let

$$\omega_{k,m}^{(*)+} = \min\{l \geqslant k : S_{k,m}^{(*)}(l) = 0\} \tag{2.6}$$

$$\omega_{k,m}^{(*)-} = \max\{l \leqslant 0 : S_{k,m}^{(*)}(l) = 0\} \tag{2.7}$$

In plain words: $k - \omega_{k,m}^{(*)+}$ and $k - \omega_{k,m}^{(*)-} - 1$ are the leftmost, respectively rightmost, sites visited by the stopped walk $\underline{X}_0^{T_{k,m}^{(*)}}$. The following theorem describes the precise asymptotics of the *process* S, properly scaled:

Theorem 1. Let $x \in [0, \infty)$, $h > 0$, and $* = L$ or R. We have

$$\left(\frac{\omega_{[Ax],[\sqrt{A}\sigma h]}^{(*)-}}{A}, \frac{\omega_{[Ax],[\sqrt{A}\sigma h]}^{(*)+}}{A}, \frac{S_{[Ax],[\sqrt{A}\sigma h]}^{(*)}([Ay])}{2\sigma\sqrt{A}} : \frac{\omega_{[Ax],[\sqrt{A}\sigma h]}^{(*)-}}{A} \leqslant y \leqslant \frac{\omega_{[Ax],[\sqrt{A}\sigma h]}^{(*)+}}{A} \right)$$

$$\Rightarrow (\omega_0^-, \omega_x^+, |W_y| : \omega_0^- \leqslant y \leqslant \omega_x^+ \,\|\, |W_0| = h) \tag{2.8}$$

in $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty, \infty)$ as $A \to \infty$. The positive constant σ is given by

$$\sigma^2 = \frac{\sum_{z \in \mathbb{Z}} z^2 \exp\{-gz^2\}}{\sum_{z \in \mathbb{Z}} \exp\{-gz^2\}} \tag{2.9}$$

(For the notion of convergence in distribution in the function space involved we refer the reader to the standard literature.[3,8])

The content of the theorem is pictorially represented on Fig. 1. It is instructive to compare this behavior with that of a simple symmetric random walk on $\mathbb{Z}$. The analogous result for the simple symmetric random walk is formulated in the Ray–Knight theory of local times (see, e.g., ref. 11). According to these classical theorems, in the ordinary random walk case the proper scaling is

$$\frac{S^{(*)}_{[Ax],[Ah]}([Ay])}{2A}$$

and the limiting process is graphically represented in Fig. 2. In conclusion both the scaling and the limiting process are different in our self-repelling case.

An immediate corollary of the previous theorem is the following limit law for the hitting times defined in (2.3), (2.4):

Corollary 1. For $x \in [0, \infty)$, $h \geqslant 0$, and $* = L$ or R, we have

$$\frac{T^{(*)}_{[Ax],[\sqrt{A}\,\sigma h]}}{2\sigma A^{3/2}} \Rightarrow (T_x \,\|\, |W_0| = h) \tag{2.10}$$

as $A \to \infty$.

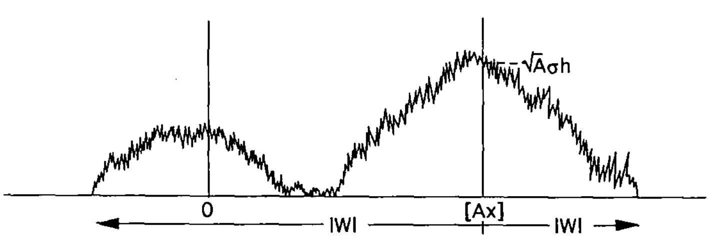

Fig. 1. The local time process $S(\cdot)$ of the TSAW with $\kappa = 1$.

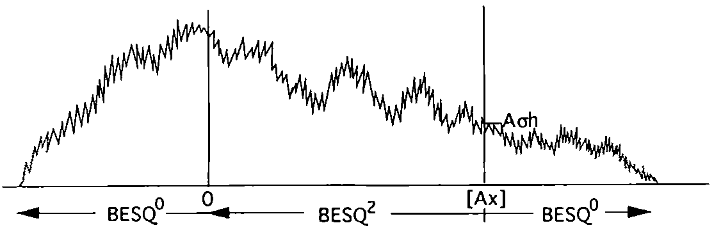

Fig. 2. The local time process $S(\cdot)$ of the simple symmetric random walk. (BESQ$^\delta$ = squared Bessel process of dimension δ).

2.2. A By-Product: A New Identity Concerning Brownian Excursions and Bessel Bridges

Our second result can be formulated in terms of Brownian motion, without any reference to the true self-avoiding walk: it is an apparently new identity concerning Brownian excursions and Bessel bridges.

For any initial condition $|W_0| = h$, the T_x defined in (2.2) clearly has an absolutely continuous distribution. Let

$$\varrho(t, x, h) = \frac{\mathbf{P}(T_x \in (t, t+dt) \,\|\, |W_0| = h)}{dt} \tag{2.11}$$

be the density of the distribution of T_x. From scaling the Brownian motion we easily get

$$b\varrho(bt, b^{2/3}x, b^{1/3}h) = \varrho(t, x, h) \tag{2.12}$$

for any $b > 0$. Define $\mathbb{R}_+ \times \mathbb{R} \ni (t, x) \mapsto \varphi(t, x) \in \mathbb{R}_+$ as follows:

$$\varphi(t, x) = \int_0^\infty \varrho\left(\frac{t}{2}, |x|, h\right) dh \tag{2.13}$$

(We shall see soon that the integral on the right-hand side is finite.) The scaling property (2.12) of ϱ implies

$$b^{2/3}\varphi(bt, b^{2/3}x) = \varphi(t, x) \tag{2.14}$$

We denote by $\hat{\varrho}$ and $\hat{\varphi}$ the Laplace transform of ϱ, respectively φ:

$$\begin{aligned}\hat{\varrho}(s, x, h) &= s\int_0^\infty e^{-st}\varrho(t, x, h)\, dt \\ &= s\mathbf{E}(e^{-sT_x} \,\|\, |W_0| = h)\end{aligned} \tag{2.15}$$

$$\hat{\varphi}(s, x) = s \int_0^\infty e^{-st} \varphi(t, x)\, dt$$

$$= \int_0^\infty \hat{\varrho}(2s, |x|, h)\, dh \tag{2.16}$$

These functions scale as follows:

$$b\hat{\varrho}(b^{-1}s, b^{2/3}x, b^{1/3}h) = \hat{\varrho}(s, x, h) \tag{2.17}$$

$$b^{2/3}\hat{\varphi}(b^{-1}s, b^{2/3}x) = \hat{\varphi}(s, x) \tag{2.18}$$

Theorem 2. Given $t \in (0, \infty)$ [respectively $s \in (0, \infty)$] fixed, $x \mapsto \varphi(t, x)$ [respectively $x \mapsto \hat{\varphi}(s, x)$] is a probability density. That is, for any $t \in (0, \infty)$ [respectively $s \in (0, \infty)$]

$$\int_{-\infty}^{\infty} \varphi(t, x)\, dx = 1 = \int_{-\infty}^{\infty} \hat{\varphi}(s, x)\, dx \tag{2.19}$$

The two assertions of (2.19) are, of course, equivalent. $\hat{\varphi}(s, \cdot)$ is the distribution $\varphi(t, \cdot)$ observed at a 'random time' of exponential distribution with mean value s^{-1}. From the scaling relations (2.14) and (2.18) one can easily see that the integrals in (2.19) do not depend on t, respectively s, and it is enough to prove (2.19) for any particular value, say $s = 1$.

2.3. Limit Theorem for the Position Process

The third result concerns the limiting distribution of the TSAW X_n for late times. We denote by $P(n, k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, the distribution of our TSAW at time n:

$$P(n, k) = \mathbf{P}(X_n = k) \tag{2.20}$$

and by $R(s, k)$, $s \in \mathbb{R}_+$, $k \in \mathbb{Z}$, the distribution of the TSAW observed at a random time θ_s, of geometric distribution

$$\mathbf{P}(\theta_s = n) = (1 - e^{-s})\, e^{-sn} \tag{2.21}$$

$$R(s, k) = (1 - e^{-s}) \sum_{n=0}^{\infty} e^{-sn} P(n, k) \tag{2.22}$$

We define the following rescaled 'densities' of the above distributions

$$\varphi_A(t, x) = A^{2/3} P([At], [A^{2/3}x]) \tag{2.23}$$

$$\hat{\varphi}_A(s, x) = A^{2/3} R(A^{-1}s, [A^{2/3}x]) \tag{2.24}$$

$t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$. It is straightforward that $\hat{\varphi}_A$ is exactly the Laplace transform of φ_A. The main result of ref. 12 was the following.

Theorem 3. For any $s \in \mathbb{R}_+$ and almost all $x \in \mathbb{R}$

$$\hat{\varphi}_A(s, x) \to \sigma^{2/3}\hat{\varphi}(s, \sigma^{2/3}x) \tag{2.25}$$

as $A \to \infty$.

This is of course a *local limit theorem* for the TSAW observed at an independent random time $\theta_{s/A}$ of geometric distribution with mean $(1 - e^{-s/A})^{-1}$. In particular, the (integral) limit law

$$\mathbf{P}(A^{-2/3}X_{\theta_{s/A}} < x) \to \int_{-\infty}^{\sigma^{2/3}x} \hat{\varphi}(s, y)\, dy \tag{2.26}$$

follows. This is a little bit short of stating the limit theorem for deterministic time

$$\mathbf{P}(A^{-2/3}X_{[At]} < x) \to \int_{-\infty}^{\sigma^{2/3}x} \varphi(t, y)\, dy \tag{2.27}$$

But, of course, we can conclude that if $X_{[At]}$ obeys any limit law as $A \to \infty$, then (2.27) also holds.

3. $\kappa \in (0, 1)$: PARTIAL RESULTS AND A CONJECTURE

Given the results presented above, it is natural to try to generalize the problem in order to get a wider range of anomalous diffusion limits, interpolating between the simple symmetric random walk and the TSAW treated in the previous section. The results and conjectures to be presented below refer to the random walk starting from the origin and governed by the law (1.2), with $\kappa \in (0, 1)$ fixed. True self-avoiding walks with $\kappa \in (1, \infty)$, respectively with power law, rather than exponential repulsion will be studied in a forthcoming paper.[13] (See also the remark after Theorem 4.)

Similarly to (2.2), we define for any $\alpha > 0$

$$T_x^{(\alpha)} = \int_{\omega_0^-}^{\omega_x^+} |W_y|^\alpha\, dy \tag{3.1}$$

We shall be particularly interested in $\alpha = 2/(\kappa + 1) \in [1, 2)$, but the definition (3.1), and Section 3.2 below make perfect sense for any positive α.

3.1. The Local Time Process and Hitting Times: A Parallel to Theorem 1

The definitions of the stopping times $T^{(\mathrm{L,R})}_{k,m}$, the local time process $S^{(\mathrm{L,R})}_{k,m}(\cdot)$, and $\omega^{(\mathrm{L,R})\pm}_{k,m}$ are the same as in (2.3)–(2.7).

The local time process of the random walk considered stopped at the hitting time $T^{(\mathrm{L,R})}_{k,m}$ obeys the following limit law:

Theorem 4. Let $\kappa \in (0, 1)$. For $x \in [0, \infty)$, $h \geqslant 0$ and $* = \mathrm{L}$ or R.

$$\left(\frac{\omega^{(*)-}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}}{A}, \frac{\omega^{(*)+}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}}{A}, \frac{S^{(*)}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}([Ay])}{2\sigma A^{1/(\kappa+1)}} : \frac{\omega^{(*)-}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}}{A} \leqslant y \leqslant \frac{\omega^{(*)+}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}}{A}\right)$$

$$\Rightarrow (\omega_0^-, \omega_x^+, |W_y|^{2/(\kappa+1)}: \omega_0^- \leqslant y \leqslant \omega_x^+ \parallel |W_0| = h^{(\kappa+1)/2}) \tag{3.2}$$

in $\mathbb{R}_- \times \mathbb{R}_+ \times D(-\infty, \infty)$ as $A \to \infty$. The positive constant σ is given by

$$\sigma^2 = \frac{(\kappa+1)^2}{2^{\kappa+2}\kappa g} \tag{3.3}$$

Remarks. One should be careful with the $\kappa \to 0$ and $\kappa \to 1$ limits. The $\kappa \to 0$ limit formally leads us back to the simple symmetric random walk, but the variance σ^2 in (3.3) explodes. The correct limit to take is $\kappa \to 0$, $g \to \infty$, $\kappa g \to a \in (0, \infty)$; this leads to *power law* rather than subexponential bond repulsion: indeed in this limit

$$\exp\{-g(n^\kappa - m^\kappa)\} \to \frac{n^{-a}}{m^{-a}}, \qquad n \neq 0 \neq m \tag{3.4}$$

True self-avoiding walks with power law repulsion will be studied in a forthcoming paper.[13] The $\kappa \to 1$ limit formally leads us back to the setup of Section 2, but the limiting variance, $\sigma^2 \to (2g)^{-1}$, differs from the variance found in Theorem 1, (2.9). The discrepancy is caused by interchanging two limiting procedures. For more details on this see the outline of the proof in the next section, especially the remark after (4.13).

Theorem 4 has an immediate corollary, too:

Corollary 2. Given the setup of Theorem 4, the asymptotics of the hitting times defined in (2.3), (2.4) is

$$\frac{T^{(*)}_{[Ax],[A^{1/(\kappa+1)}\sigma h]}}{2\sigma A^{(\kappa+2)/(\kappa+1)}} \Rightarrow (T_x^{(2/(\kappa+1))} \parallel |W_0| = h^{(\kappa+1)/2}) \tag{3.5}$$

as $A \to \infty$, with σ given by (3.3).

3.2. Generalization of Theorem 2: A Conjecture

Given Theorem 4 and the techniques of proof of Theorem 3 (see also Section 4.3 below), it is easy *to guess* the limiting distribution of the properly scaled $A^{-(\kappa+1)/(\kappa+2)}X_{[At]}$. Let $\varrho^{(\alpha)}$ be the distribution density of $T^{(\alpha)}$ defined in (3.1):

$$\varrho^{(\alpha)}(t, x, h) = \frac{\mathbf{P}(T_x^{(\alpha)} \in (t, t+dt) \,\|\, |W_0|^\alpha = h)}{dt} \tag{3.6}$$

and similarly to (2.13) we define

$$\varphi^{(\alpha)}(t, x) = \int_0^\infty \varrho^{(\alpha)}\left(\frac{t}{2}, |x|, h\right) dh \tag{3.7}$$

$\hat{\varrho}^{(\alpha)}$ and $\hat{\varphi}^{(\alpha)}$ are defined by the Laplace transforms (2.15) and (2.16). These definitions are of course completely independent of the TSAW problem and make perfect sense for *any* $\alpha \in (0, \infty)$. The scaling laws of these functions are

$$b\varrho^{(\alpha)}(bt, b^{2/(\alpha+2)}x, b^{\alpha/(\alpha+2)}h) = \varrho^{(\alpha)}(t, x, h) \tag{3.8}$$

$$b^{2/(\alpha+2)}\varphi^{(\alpha)}(bt, b^{2/(\alpha+2)}x) = \varphi^{(\alpha)}(t, x) \tag{3.9}$$

$$b\hat{\varrho}^{(\alpha)}(b^{-1}s, b^{2/(\alpha+2)}x, b^{\alpha/(\alpha+2)}h) = \hat{\varrho}^{(\alpha)}(s, x, h) \tag{3.10}$$

$$b^{2/(\alpha+2)}\hat{\varphi}^{(\alpha)}(b^{-1}s, b^{2/(\alpha+2)}x) = \hat{\varphi}^{(\alpha)}(s, x) \tag{3.11}$$

Conjecture. Let $\alpha \in (0, \infty)$. Given $t \in (0, \infty)$ [respectively $s \in (0, \infty)$] fixed, $x \mapsto \varphi^{(\alpha)}(t, x)$ [respectively $x \mapsto \hat{\varphi}^{(\alpha)}(s, x)$] is a probability density. That is, for any $t \in (0, \infty)$ [respectively $s \in (0, \infty)$]

$$\int_{-\infty}^{\infty} \varphi^{(\alpha)}(t, x)\, dx = 1 = \int_{-\infty}^{\infty} \hat{\varphi}^{(\alpha)}(s, x)\, dx \tag{3.12}$$

The integrals are not greater than 1; this follows from (4.34); see also the Remark after Theorem 5.

The motivation of the conjecture is of course the fact that, for $\alpha = 2/(\kappa+1) \in (1, 2)$, $\hat{\varphi}^{(\alpha)}(t, x)$ turns out to be the asymptotic distribution density of $A^{-(\kappa+1)/(\kappa+2)}X_{[At]}$ as $A \to \infty$. For $\alpha = 1$, Theorem 2 establishes the conjecture. As we shall see in the next section, the integral on the right-hand side of (3.12) can be expressed in terms of expectations of certain functionals of the three-dimensional Bessel bridge. The expressions found make sense in the limits $\alpha \to 0$ and $\alpha \to \infty$, too. And in these limits we can prove the assertion. Further on, for the value $\alpha = 2$ we get rather explicit expressions in terms of some hypergeometric integrals. We were not able to *prove* the equality in this case, but *numerical* integration gives a surprisingly accurate good result.

3.3. Limit Theorem for the Position Process

The distributions of X_n and X_{θ_s} are again denoted by $P(n, k)$, respectively $R(s, k)$ [see (2.20)–(2.22)]. The properly rescaled 'densities' of the above distributions are now

$$\varphi_A(t, x) = A^{(\kappa+1)/(\kappa+2)} P([At], [A^{(\kappa+1)/(\kappa+2)} x]) \tag{3.13}$$

$$\hat{\varphi}_A(s, x) = A^{(\kappa+1)/(\kappa+2)} R(A^{-1}s, [A^{(\kappa+1)/(\kappa+2)} x]) \tag{3.14}$$

$t, s \in \mathbb{R}_+$, $x \in \mathbb{R}$. The limit theorem for the position process is formulated conditionally, relying on the validity of the Conjecture:

Theorem 5. Assume that the Conjecture formulated in the previous subsection holds. For any $s \in \mathbb{R}_+$ and almost all $x \in \mathbb{R}$

$$\hat{\varphi}_A(s, x) \to \sigma^{(\kappa+1)/(\kappa+2)} \hat{\varphi}^{(2/(\kappa+1))}(s, \sigma^{(\kappa+1)/(\kappa+2)} x) \tag{3.15}$$

as $A \to \infty$, with σ given in (3.3).

Remark. Without assuming the validity of the conjecture, we get the *inequality* (4.34). That inequality further implies that the integrals in (3.12) are not greater than 1.

The integral version of this local limit theorem reads:

$$\mathbf{P}(A^{-(\kappa+1)/(\kappa+2)} X_{\theta_{s/A}} < x) \to \int_{-\infty}^{\sigma^{(\kappa+1)/(\kappa+2)} x} \hat{\varphi}(s, y)\, dy \tag{3.16}$$

And again we may conclude that if $X_{[At]}$ obeys any limit law as $A \to \infty$, then

$$\mathbf{P}(A^{-(\kappa+1)/(\kappa+2)} X_{[At]} < x) \to \int_{-\infty}^{\sigma^{(\kappa+1)/(\kappa+2)} x} \varphi(t, y)\, dy \tag{3.17}$$

4. FUNDAMENTAL IDEAS OF PROOFS

We sketch the proofs of the assertions made in the previous section. The numbers of the subsections correspond to those of Section 3.

4.1. For sake of simplicity we outline the main ideas of the proof of Theorem 4 for the case $h = 0$ and $* = L$, i.e., we stop the random walk at the first arrival at a distant site $k \in \mathbb{Z}$. In the end we shall take $k = [Ax]$

with $x>0$ and $A \to \infty$. With a little abuse of notation we denote by $T_k = T_{k,0}^{(\mathrm{L})}$ the first hitting time of the site $k \in \mathbb{Z} \cap (0, \infty)$. Define

$$L_k(l) = \begin{cases} \frac{1}{2}v(k-l \mid \underline{X}_0^{T_k}) - \frac{1}{2} & \text{if } \ 0 \leqslant l < k \\ \frac{1}{2}v(k-l \mid \underline{X}_0^{T_k}) & \text{if } \ k \leqslant l \end{cases} \tag{4.1}$$

In plain words: $L_k(l)$ is the number of left steps $(k-l) \to (k-l-1)$ performed by the random walk $\underline{X}_0^{T_k}$, stopped at the first hitting of the site k, and it is essentially half of the local time spent on the edge $(k-l-1, k-l)$.

The clue of the proof is the observation that, with k fixed, the local time process $L_k(\cdot)$ defined in (4.1) is a Markov chain on the state space $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. Apparently this trick has its origin in Knight's paper[6] and it is the cornerstone of Ray–Knight theory of the local time of simple symmetric random walk and Brownian motion. The same sort of trick was applied to the study of random walks in random environments in ref. 5. However, as opposed to the previous applications of this trick, the Markov process arising in our case will be more complicated than a branching process[6] or a branching process with random offspring distribution.[5]

A finite walk which hits k for the first time at time i

$$0 = x_0, x_1, \ldots, x_{i-1}, x_i = k, \qquad \min\{j\colon x_j = k\} = i \tag{4.2}$$

determines uniquely a finite sequence

$$\begin{aligned} &\Lambda(0) = 0 \\ &\underline{\lambda}(l) = \begin{cases} (\lambda_0(l), \ldots, \lambda_{\Lambda(l-1)}(l)) & \text{if } \ 1 \leqslant l \leqslant k \\ (\lambda_1(l), \ldots, \lambda_{\Lambda(l-1)}(l)) & \text{if } \ l > k \end{cases} \\ &\Lambda(l) = \sum_{p=0(1)}^{\Lambda(l-1)} \lambda_p(l) \end{aligned} \tag{4.3}$$

where $\lambda_p(l)$ is the number of steps $(k-l) \to (k-l-1)$ between the pth and $(p+1)$th steps $(k-l+1) \to (k-l)$ performed by the walk. We shall refer to the sequence (4.3) as the system of left steps of the walk (4.2). Finiteness of the system of left steps means that there exists an $l_{\max} > k$ such that $\Lambda(l_{\max}) = 0$. This correspondence between finite walks hitting k and finite systems of left steps is *one-to-one*: given the sequence (4.3), one can reconstruct the complete walk (4.2) univocally.

Let (4.2) be a finite walk hitting k and (4.3) the corresponding system of left steps. Since T_k is a stopping time, the probability that the TSAW coincides with (4.2) till T_k is

$$\mathbf{P}(\underline{X}_0^{T_n} = \underline{x}_0^i) = \prod_{j=1}^{i} w_j(\underline{x}_0^j) \tag{4.4}$$

where, according to (1.2), the weight of the jth step is

$$w_j(\underline{x}_0^j) = \frac{\exp\{-gv^\kappa([(x_{j-1}+x_j+1)/2] \mid \underline{x}_0^{j-1})\}}{\exp\{-gv^\kappa(x_{j-1}+1 \mid \underline{x}_0^{j-1})\} + \exp\{-gv^\kappa(x_{j-1} \mid \underline{x}_0^{j-1})\}} \tag{4.5}$$

Now, rearranging the product in (4.4), we get

$$P(\underline{X}_0^{T_n} = \underline{x}_0^i) = \prod_{l \geq 0} \left\{ \prod_{0 \leqslant j \leqslant i:\, x_{j-1} = k-l} w_j(\underline{x}_0^j) \right\} \tag{4.6}$$

This last expression can be written in terms of the 'system of left steps' (4.3). A careful transcription yields

$$\begin{aligned} \mathbf{P}(\underline{X}_0^{T_k} = \underline{x}_0^i) = {} & \left[\prod_{l=1}^{k-1} \mathscr{P}_-(\Lambda(l-1); \underline{\lambda}(l)) \right] \\ & \times \mathscr{P}_0(\Lambda(k-1); \underline{\lambda}(k)) \left[\prod_{l>k} \mathscr{P}_+(\Lambda(l-1); \underline{\lambda}(l)) \right] \end{aligned} \tag{4.7}$$

From (4.7) one can see that the process $L_k(\cdot)$ is indeed a *Markov chain* on the state space $\mathbb{Z}_+$, homogeneous in the intervals $0 \leqslant l < k$, $l = k$, and $l > k$.

Remark. In case of the TSAW with *site repulsion* this rearrangement of the product, i.e., the transcription of (4.4) to (4.7), cannot be performed. This is the step where the proof of similar results for the site repulsive TSAW fails.

The explicit expressions of $\mathscr{P}_{\pm,0}$ are rather complicated and not particularly instructive. More transparent is the following generalized Pólya urn description: Let $0 < l < k$ be fixed. [For $l \geqslant k$ the forthcoming argument also applies with one minor change; see the comment after (4.8).] Consider the succession of those steps of the walk $\underline{X}_0^{T_k}$ which start from the lattice site $k - l$. These determine a sequence s_i, $i = 1, 2,\ldots,$ of ± 1's according to the direction of the ith step starting from that site. This sequence follows a generalized Pólya urn scheme:

$$\begin{aligned} & \mathbf{P}(s_{i+1} = \mp 1 \,\|\, \#\{1 \leqslant j \leqslant i : s_j = -1\} = p \wedge \#\{1 \leqslant j \leqslant i : s_j = +1\} = r) \\ & \quad = \tfrac{1}{2} \mp \tfrac{1}{2} \tanh\{2^{\kappa-1} g((p+\tfrac{1}{2})^\kappa - r^\kappa)\} \end{aligned} \tag{4.8}$$

and the urn schemes based at different lattice sites are independent. [For $l > k$ instead of $(p + 1/2)^\kappa$ (respectively r^κ) we get p^κ (respectively $(r + 1/2)^\kappa$) in the previous formula, but this does not influence the forthcoming argument.] Let us consider an urn scheme defined by (4.8) and define

$$\tau_0 = 0, \qquad \tau_{n+1} = \min\{j > \tau_n : s_j = +1\} \tag{4.9}$$

$$Z_n = \tau_{n+1} - (n+1) \tag{4.10}$$

In plain words: Z_n is the number of -1's appearing before the $(n+1)$th $+1$ appears. We claim that

$$Y_n = Z_n^{(\kappa+1)/2} - n^{(\kappa+1)/2} \tag{4.11}$$

converges in distribution to a Gaussian random variable of mean zero and variance σ^2 given in (3.3) as $n \to \infty$. Assume as an Ansatz that $Y_n = O(1)$ as $n \to \infty$. We denote

$$\zeta_{n+1} = Z_{n+1} - Z_n \tag{4.12}$$

Given (4.8), we see that, for large values of n, ζ_{n+1} is a random variable of expectation and variance

$$\mathbf{E}\zeta_{n+1} = 1 - \frac{2^{\kappa+1}\kappa g}{\kappa+1} Y_n n^{(\kappa-1)/2} + O(n^{\kappa-1}), \qquad \mathbf{D}^2\zeta_n = 2 + O(n^{(\kappa-1)/2}) \tag{4.13}$$

Remark. We can see here and in the forthcoming asymptotic expansions why the case $\kappa = 1$ is not recovered when taking $\kappa \to 1$ *after* proving the limit theorem: if $\kappa = 1$, then error terms in (4.13) become of the same order as the relevant ones.

In order to prove our assertion, we write the increments of the process Y_n:

$$\begin{aligned} Y_{n+1} - Y_n &= (Z_{n+1}^{(\kappa+1)/2} - Z_n^{(\kappa+1)/2}) - ((n+1)^{(\kappa+1)/2} - n^{(\kappa+1)/2}) \\ &= (n^{(\kappa+1)/2} + Y_n)\left(\left(1 + \frac{\zeta_{n+1}}{(n^{(\kappa+1)/2} + Y_n)^{2/(\kappa+1)}}\right)^{(\kappa+1)/2} - 1\right) \\ &\quad - n^{(\kappa+1)/2}\left(\left(1 + \frac{1}{n}\right)^{(\kappa+1)/2} - 1\right) \end{aligned} \tag{4.14}$$

If $\kappa \in (0, 1)$, careful analysis of this last identity yields, to leading orders, the following expressions of the conditional expectation and conditional variance of $Y_{n+1} - Y_n$:

$$\mathbf{E}(Y_{n+1} - Y_n \,\|\, Y_n) = -2^{\kappa}\kappa g Y_n n^{\kappa-1} + O(n^{-1}) \tag{4.15}$$

$$\mathbf{D}^2(Y_{n+1} - Y_n \,\|\, Y_n) = \frac{(\kappa+1)^2}{2} n^{\kappa-1} + O(n^{-1}) \tag{4.16}$$

Since $n^{\kappa-1} \to 0$, these last relations asymptotically become the stochastic differential equation of the Ornstein–Uhlenbeck process,

$$dY_t = -2^\kappa \kappa g Y_t \, dt + \frac{(\kappa+1)}{\sqrt{2}} dw_t, \tag{4.17}$$

and since $\sum_{n=1}^{\infty} n^{\kappa-1} = \infty$, the distribution of Y_n will converge to the stationary distribution of Y_t defined by (4.17), which is the Gaussian of zero expectation and variance σ^2 given in (3.3). Actually we can control the rate of this convergence: the variational distance of the distribution of Y_n from that of the Gaussian of variance σ^2 is dominated by $C_1 \exp(-C_2 n^{(\kappa-\varepsilon)})$. This follows from the fact that (a) the Ornstein–Uhlenbeck process (4.17) converges to its equilibrium distribution exponentially fast in t and (b) $\sum_1^n m^{\kappa-1} \approx (1/\kappa) n^\kappa$.

The argument outlined above suggests that it is more natural to consider the Markov chain $L_k^{(\kappa+1)/2}(\cdot)$ (on the state space $\{n^{(\kappa+1)/2} \mid n \in \mathbb{Z}_+\} \subset \mathbb{R}_+$) than $L_k(\cdot)$. We define

$$\xi_k(l) = L_k^{(\kappa+1)/2}(l) - L_k^{(\kappa+1)/2}(l-1), \qquad l \geq 1 \tag{4.18}$$

The above argument tells us that the conditional distribution of the step $\xi_k(l)$, given that the previous position was large, $L_k^{(\kappa+1)/2}(l-1) \gg 1$, is close [closer than $C_1 \exp(-C_2 x^{(2\kappa-\varepsilon)/(\kappa+1)})$ in variation distance] to a fixed Gaussian distribution ϕ. Taking $k=[Ax]$, $A \to \infty$, we can couple the Markov chain $L_{[Ax]}^{(\kappa+1)/2}(\cdot)$ with a reflected random walk on $\mathbb{R}_+$ with homogeneous step distribution ϕ in such a way that their sup-distance is $o(A^{1/2})$ with probability converging to 1. This coupling argument is identical to the one in the proof of Theorem 1 in ref. 12. We do not repeat the lengthy technical parts here. Eventually we get

$$\frac{L_{[Ax]}^{(\kappa+1)/2}([Ay])}{\sqrt{A}\,\sigma} \Rightarrow |W_y| \tag{4.19}$$

which is equivalent to (3.2), with $h=0$.

4.2. Using Itô's and Bismut's characterizations of Brownian excursion measure (see refs. 4 and 11), we can express the integral on the right-hand side of (3.12) (with the choice $s=1$) in terms of expectations of certain functionals of the three-dimensional Bessel bridge. Let B_t, $0 \leq t \leq 1$, be a standard three-dimensional Bessel bridge over the time interval [0, 1], and for $0 \leq \beta < \infty$ denote

$$\tau_\beta = \int_0^1 B_t^\beta \, dt \tag{4.20}$$

Repeating the first steps of the proof of Theorem 2 in ref. 12, we can rewrite our Conjecture as

$$\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\alpha+2}\left\{\mathbf{E}(\tau_{\alpha-1}\tau_\alpha^{-1})+\frac{2}{\alpha+2}\frac{\Gamma(\alpha/(\alpha+2))\,\Gamma(1/(\alpha+2))}{\Gamma((\alpha+1)/(\alpha+2))}\right.$$
$$\left.\times\frac{\mathbf{E}(\tau_\alpha^{-1/(\alpha+2)})}{\mathbf{E}(\tau_\alpha^{1/(\alpha+2)})}\mathbf{E}(\tau_{\alpha-1}\tau_\alpha^{-\alpha(\alpha+2)})\right\}\stackrel{?}{=}1 \tag{4.21}$$

By use of a theorem of Biane and Yor (see ref. 2 and Theorem XI.3.5 in ref. 11) $\mathbf{E}(\tau_\alpha^{-1/(\alpha+2)})$ and $\mathbf{E}(\tau_\alpha^{1/(\alpha+2)})$ can be expressed explicitly for any α, and consequently we can reduce our conjecture to

$$\frac{1}{\sqrt{2\pi}}\frac{\alpha}{\alpha+2}\left\{\mathbf{E}(\tau_{\alpha-1}\tau_\alpha^{-1})+\frac{2^{(2\alpha+2)/(\alpha+2)}}{(\alpha+2)^{2\alpha/(\alpha+2)}}\right.$$
$$\left.\times\frac{\Gamma(\alpha/(\alpha+2))\,\Gamma(3/(\alpha+2))\,\Gamma^2(2/(\alpha+2))}{\Gamma((\alpha+1)/(\alpha+2))\,\Gamma(4/(\alpha+2))}\mathbf{E}(\tau_{\alpha-1}\tau_\alpha^{-\alpha/(\alpha+2)})\right\}\stackrel{?}{=}1 \tag{4.22}$$

For $\alpha=1$ the equality was established in ref. 12. It is interesting to note that although the expressions were *a priori* defined for $0<\alpha<\infty$, they still make sense in the limits $\alpha\to 0$ and $\alpha\to\infty$. We have

$$\lim_{\alpha\to 0}(\text{l.h.s. of (4.22)})=\frac{\mathbf{E}(\tau_{-1})}{(2\pi)^{1/2}} \tag{4.23}$$

$$\lim_{\alpha\to\infty}(\text{l.h.s. of (4.22)})=\frac{1}{(2\pi)^{1/2}}\left(\mathbf{E}\left(\frac{1}{m}\right)+\frac{4}{3}\mathbf{E}(m)\right) \tag{4.24}$$

where

$$m=\sup_{0\leqslant t\leqslant 1}B_t \tag{4.25}$$

The expectations appearing in (4.23) and (4.24) are known:[2,11]

$$\mathbf{E}(\tau_{-1})=(2\pi)^{1/2},\qquad \mathbf{E}(m)=\left(\frac{\pi}{2}\right)^{1/2},\qquad \mathbf{E}\left(\frac{1}{m}\right)=\frac{(2\pi)^{1/2}}{3} \tag{4.26}$$

Plugging these expressions into (4.23) and (4.24), we find that the Conjecture holds at least in these limiting cases. Another interesting case is $\alpha=2$: using Theorem XI.3.2 and its Corollary XI.3.3 ref. 11, we can get a more explicit formula:

$$(\text{l.h.s. of (4.22)})|_{\alpha=2}=\int_0^\infty\int_0^\infty\frac{(\sinh s\,\sinh t)^{1/2}}{\sinh^2(s+t)}\,ds\,dt$$
$$+\frac{1}{\pi}\int_0^\infty\int_0^\infty\frac{(\sinh s\,\sinh t)^{1/2}}{\sinh^2(s+t)}(s+t)\,ds\,dt \tag{4.27}$$

We have not been able to evaluate analytically these expressions, but numerical integration using the *Mathematica* software package yields very accurate agreement with our conjecture.

4.3. To prove Theorem 5 we note first that

$$P(n,k)=\mathbf{P}(X_n=k)=\sum_{m=0}^{\infty}\left[\mathbf{P}(T^{(\mathrm{L})}_{k,m}=n)+\mathbf{P}(T^{(\mathrm{R})}_{k,m}=n)\right] \tag{4.28}$$

On the other hand, from the definition (3.14) of $\hat{\varphi}_A$,

$$\hat{\varphi}_A(s,x)=\frac{1-e^{-s/A}}{s/A}sA^{-1/(\kappa+2)}\sum_{n=0}^{\infty}e^{-ns/A}P(n,[A^{(\kappa+1)/(\kappa+2)}x]) \tag{4.29}$$

Combining (4.28) and (4.29), we are led to

$$\begin{aligned}\hat{\varphi}_A(s,x)=&\frac{1-e^{-s/A}}{s/A}\\ &\times sA^{-1/(\kappa+2)}\sum_{m=0}^{\infty}\left[\mathbf{E}(\exp(-sT^{(\mathrm{L})}_{[A^{(\kappa+1)/(\kappa+2)}x],m}/A))\right.\\ &\left.+\mathbf{E}(\exp(-sT^{(\mathrm{R})}_{[A^{(\kappa+1)/(\kappa+2)}x],m}/A))\right]\end{aligned} \tag{4.30}$$

Defining

$$\hat{\varrho}^{(\mathrm{L},R)}_A(s,x,h)=s\mathbf{E}(\exp\{-sT^{(\mathrm{L},\mathrm{R})}_{[A^{(\kappa+1)/(\kappa+2)}x],[A^{1/(\kappa+2)}\sigma h]}/(2\sigma A)\}) \tag{4.31}$$

we find that (4.30) reads

$$\hat{\varphi}_A(s,x)=\frac{1-e^{-s/A}}{s/A}\frac{1}{2}\int_0^{\infty}(\hat{\varrho}^{(\mathrm{L})}_A(2\sigma s,x,h)+\hat{\varrho}^{(\mathrm{R})}_A(2\sigma s,x,h))\,dh \tag{4.32}$$

From Corollary 2 it follows that for any $s>0$, $x\in[0,\infty)$, and $h>0$

$$\hat{\varrho}^{(\mathrm{L},\mathrm{R})}_A(s,x,h)\to\hat{\varrho}^{(2/(\kappa+1))}(s,x,h) \tag{4.33}$$

as $A\to\infty$. Conditions (4.32) and (4.33) imply

$$\begin{aligned}\liminf_{A\to\infty}\hat{\varphi}_A(s,x)\geqslant&\int_0^{\infty}\hat{\varrho}^{(2/(\kappa+1))}(2\sigma s,|x|,h)\,dh\\ &=\sigma^{(\kappa+1)/(\kappa+2)}\hat{\varphi}^{(2/(\kappa+1))}(s,\sigma^{(\kappa+1)/(\kappa+2)}x)\end{aligned} \tag{4.34}$$

On the other hand, assuming the validity of the Conjecture, we have

$$\int_{-\infty}^{\infty}\hat{\varphi}_A(s,x)\,dx=1=\int_{-\infty}^{\infty}\hat{\varphi}^{(2/(\kappa+1))}(s,x)\,dx \tag{4.35}$$

The statement of Theorem 5 follows from (4.34) and (4.35).

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NOTE ADDED IN PROOF

The Conjecture formulated in subsection 3.2. has been proved. Its proof will appear in ref. 13.

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